

INCREASING RETURNS TO SCALE, PRICE DISPERSION, AND THE DISTRIBUTION OF RETURNS TO INNOVATION

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Models of endogenous growth have not been able to account for the variety of empirically observed distributional properties of the returns to innovation, in part, because of the limitations necessarily imposed on competition to cope with increasing returns to scale. Exponential growth, fat tails, Pareto–Levy distributed upper tails, and upper value outliers, are associated with increasing returns to scale and innovation. At the same time, properties such as bifurcated research investment strategies, bimodal returns to innovation, and Laplace distributed firm growth rates are products of competition. We build an agent-based model of endogenous technical change in which heterogeneous investments in patented knowledge and increasing returns to scale emerge these distributional properties within a competitive market. The combination of heterogeneous agents, costly information, and patents allow for a competitive landscape to persist amidst increasing returns. The ability of model to foster a coexistence of competition and increasing returns underlies the observed distributional properties.

Keywords: Patents, Endogenous Growth, Increasing Returns to Scale, Price Dispersion, Search, Heterogeneous Agents

1. INTRODUCTION

Theories of endogenous technical change built with knowledge serving as a non-rival input into productivity and, in turn, as a source of increasing returns to scale, have served to model exponential growth [Grossman and Helpman (1994); Romer (1994); Warsh (2006)]. The capacity to cope with increasing returns to scale, however, motivated the abandonment of price-taking perfect competition and the allowance of market power within firms [Romer (1990); Grossman and

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Helpman (1991); Aghion and Howitt (1992)]. It should not be surprising, given this reliance on knowledge inputs and market power, that intellectual property rights, or patents, have become a major topic of exploration in theories of endogenous growth [Horowitz and Lai (1996); Futagami and Iwaisako (2003); O’Donoghue and Zweimüller (2004); Iwaisako and Futagami (2007); Chu and Pan (2013)].¹ This exploration of endogenous growth and intellectual property rights, however, has been limited in important ways by reliance on traditional monopolistic competition.

Models incorporating patents into theories of endogenous growth have not, to date, been able to reproduce the variety of peculiar distributional stylized facts associated with returns to innovation.² These stylized facts include (i) bimodality [Gans and Quiggin (2003); Demirel and Mazzucato (2008); Dosi and Nelson (2010)], (ii) large kurtosis (“fat tails”) [Demirel and Mazzucato (2008)], and (iii) Pareto–Levy distributed upper tails [Epstein and Wang (1994); Axtell (1999); Scherer et al. (2000); Silverberg and Verspagen (2007)]. We offer an alternative modeling strategy that allows for endogenous technical change, is characterized by long-run increasing returns to scale, and remains competitive³ in the long run. Emergent from the actions taken by autonomous, heterogeneous agents are mixture distributions of market returns that are bimodal and fat-tailed, including Pareto–Levy distributed upper tails. Further, the emergent distributions are dominated by a handful of extreme values, as is often observed in the data. This peculiar mixture distribution is similar to those observed in patent revenue return research, particularly returns in research-intensive industries, such as pharmaceuticals and others with persistent patenting [Dosi (2007)].⁴

Patents bring the necessary market power to firms that seek to obtain monopoly rents from their excludable private knowledge. This excludable private knowledge, however, also engenders heterogeneity across firms that are all producing with differing knowledge inputs. Heterogeneous knowledge quickly leads to heterogeneity in productive capacity, marginal products of standard (rival) inputs, and prices. Such a world is considerably less tractable for traditional modeling, and is typically inhospitable to decentralized competition. Implementing a form of monopolistic competition within the model in the form of a continuum of goods produced by firms, à la Romer (1990), returns us to more tractable territory, but this model structure comes at a cost. With a continuum of goods in demand, and each firm producing a unique good that cannot be perfectly substituted for with goods produced by competing firms, our creative landscape is now considerably less destructive. Imperfect substitution, long thought to be necessary to allow many firms to exist in an industry with increasing returns, attenuates the consequences of discoveries that would be explosive in a world with perfect substitutes, rapidly leading to market domination by a small number of firms. The monopolistic competition model, governed by the Law of One Price, retains the representative firm by allowing heterogeneous goods. We provide a model using the exact opposite: a set of heterogeneous firms competing to produce and sell a single homogenous good, each offering the good to consumers at its own distinct price.⁵

There is considerable evidence that the returns to research are highly skewed, with distributions dominated by extreme values. Research into these returns has used a variety of creative datasets, including citation records, initial public stock offerings (IPOs), and self-reported revenue returns to patents [Harhoff et al. (1998, 1999)]. The most appropriate statistical distribution for the characterization of the returns seems to be some combination of the lognormal and Pareto–Levy distributions [Scherer et al. (2000); Silverberg and Verspagen (2007)]. Power law distributions have been found in market competition and concentration data. Axtell (2001b) finds that the size of firms, in terms of individuals employed, is Zipf distributed in the United States. Within power law distributions, the upper tail accounts for an extraordinary share of the distribution’s value. Models that account for the growth derivative of technical innovation that leverage some form of market power stand to benefit from either including such features of the returns to research or, preferably, generating them endogenously [Luttmer (2007)]. Concerns about the importance of the distribution of research outcomes, in particular of the upper tail and outliers, have been expressed recently [Silverberg and Verspagen (2007)]. They note that such Pareto power law distributions might not even have first moments, something that has severe implications for risk analysis. The matter is only further complicated by the empirical observation of bimodality—the separation of dominant and “fringe” firms [Bottazzi et al. (2007); Dosi (2007)], most notably in the pharmaceutical industry [Demirel and Mazzucato (2008)]. What this adds up to is an especially unusual outcome to that we are seeking to replicate—a distribution that is bimodal, largely lognormal, with a Pareto–Levy upper tail, and a handful of extreme values that dominate the distribution.

The aims of this paper are threefold. First, we seek to build a model characterized by the increasing returns to scale and exponential growth properties of existing models of endogenous technical change and growth. Second, we abandon the traditional monopolistic competition model and replace it with a model of competitive firms producing a homogenous good in a market characterized by price dispersion. Third, we simulate the model under a variety of parameterizations and examine the distributional properties of returns to investment in research. It is our hypothesis that combining increasing returns to scale and competition amongst firms engaged in an innovative race will replicate a greater number of the stylized facts that were previously disparate across models in the literature.

2. AN AGENT-BASED COMPUTATIONAL MODEL

Our agent-based model is composed of a population of heterogeneous, individually autonomous households and firms that make decisions in accordance with their type, unique information set, and personal history, and the rules that govern their behavior. This methodology offers two overarching advantages. First, it allows knowledge to be truly dispersed, with each agent holding a unique subset of the information available in the market. Second, the deep population heterogeneity⁶ of the model allows the exploration of emergent distributions whose properties we

can observe not just in terms of first and second moments, but as entire populations of values [Epstein and Axtell (1996); Epstein (2006); Lux (2006); Alfarano and Lux (2007); Gatti et al. (2007)]. This is especially important given our interest in the upper tails and extreme values.

Our model is built using elements prominent in O'Donoghue and Zweimüller (2004) and Iwaisako and Futagami (2007), and more generally in agent-based models of technical change [Dawid (2006); Dosi et al. (2010)]. Like the model presented in O'Donoghue and Zweimüller (2004), our model is composed of two sectors, one in which technology investment and innovation are possible and one in which innovation is not possible, with inputs of only labor and capital. Individual technology-enabled firms produce a homogenous quality primary good (q), whereas an aggregated nontechnical sector (NTS) produces a secondary good (x). The full labor market is exogenous to the model. Households earn a wage that is drawn from a lognormal distribution parameterized by the marginal product of labor within firms and maximize a universal utility function by purchasing a combination of x and q . Although the supply of labor is not limited, the cost of labor is always increasing with its marginal product. We will see that growth within the model is not dependent on the exogenous increase in labor.⁷

Time in the model occurs in discrete steps and substeps. Sets of agents (organized by type) are activated in a fixed schedule, but within each set, agents are activated in a randomized order. Although firms are effectively acting simultaneously, households are not. A household may purchase the last of a firm's inventory or fill its final hiring slot. Potential order effects add to the complexity of model outcomes, but constant randomizing of activation order prevents model artifacts [Axtell (2001a)].

Although agents, within their types, are homogenous in capacity, exogenous parameterization, and behavioral rules, they each face a world with costly, imperfect, and heterogeneous information. Households search for lower prices, seeking to maximize their consumptive bundle, although constrained by a finite amount of time to be split between wage earning and search. Firms, on the other hand, face the uncertainty of a research process that may or may not yield a competitive increase in excludable knowledge, as well as a marketplace of consumers that may or may not discover them as low price providers of goods. They respond to these uncertainties by making decisions regarding research investment predicated on simple heuristics and limited information. Given the complexity of the relationships between households and firms, the NTS is governed by a number of simplifying assumptions that grant the model additional tractability. The NTS operates as a single agent in the model, pays a single uniform wage, and always meets the sum of its market orders. Further, the NTS is governed by diminishing returns to scale.

The labor supply in the model is exogenous, commanding a wage equal to its lagged average marginal product. Capital grows as a set fraction of the gross product sum from the previous time step. Growth is driven by technical innovation. As within a Schumpeterian model of creative destruction [Aghion and Howitt

(1992)], innovation is motivated by the desire to both gain monopoly rents and avoid bankruptcy. In this manner both the carrot and the stick are applied at every step of the model: success in research and development leads to lower production costs, higher rents, and more customers, whereas failure leads to higher prices and fewer customers and brings the firm one step closer to closing its doors. The prospect of permanent failure is one of the salient features of working with a competitive market for a homogenous good. In a monopolistic competition model, where goods exist along a continuum, there is no prospect for complete failure to attract customers. This is where creative destruction is tamed in models of monopolistic competition. In our model, on the other hand, with agents searching over a set of producers offering a homogenous good, a firm with inferior productive technology will be unable to offer a competitive price and will be more likely to be passed over by potential customers. This market remains competitive,⁸ as opposed to collapsing into monopoly, because of price dispersion and costly search, which allows second-best firms to attract sufficient customers to retain positive profits, or at the very least manageable losses that can be endured in the short run [Levy and Makowsky (2010)]. Further, the expiration of patents and the subsequent sharing of previously private knowledge allows turnover in who stands as the technology leader [Grossman and Helpman (1991)]. In reality, it is not just profit, but the prospect of losses and bankruptcy that motivates investment in research and development.

In contrast to traditional general equilibrium models, there is no social planner maximizing agent utility, nor a Walrasian auctioneer finding market-clearing prices. Each agent, governed by type (firm, household)-specific rules, is autonomous. From the thousands of interacting decision-making agents emerge aggregate trends in research investment, technology, growth, profits, and market concentration. Agents are generally myopic and backward-looking and employ relatively simple search strategies. They are governed by strictly bounded rationality and costly information, but nonetheless manage to prosper in what are often rapidly growing economies.

2.1. Simulation Steps and Substep Ordering

Our model is characterized by a schedule of agent decisions and model events, which we summarize here and describe in more technical detail in the subsequent section. The schedule plays out in a series of steps and substeps.⁹ A run of the model is constituted by an initialization ($t = 0$) followed by a set number of model steps ($t = 1, \dots, T$), during which every agent is activated in random order, as arranged by the model substeps. The substeps are ordered as follows:

1. The wage rate paid by firms is set to the previous step's average marginal product of labor. Households are assigned a wage that is a random draw from a lognormal distribution parameterized by firm wage rate.
2. All expired patents are made public; the largest patent value is added to the cumulative stock of public knowledge. All subsuperior knowledge disappears.

3. Each firm conducts research.
4. The NTS sets the price for secondary goods.
5. Households are activated in random order and execute τ searches over the set of all available prices. Once they have decided on their fixed sample sizes, the first prices in their discovery sets are their sellers' prices from the previous time step. Once the search is concluded, the households maximize their utility functions, choosing optimal bundles of q and x .
6. Having received all of their market orders, firms will acquire the amounts of labor and capital necessary to produce q_j and fulfill all existing market orders.
7. If a firm is unable to turn a profit, it may go bankrupt. Bankruptcy results when a firm's outstanding debt is greater than the quantity that is available in the loan market. In the model simulations executed in this paper, firms were exempt from bankruptcy rules during the first 20 simulation steps, allowing them to adapt to initialized conditions.
8. If firms are, on the average, profitable, a new firm will enter the market. The new firm will be initiated as a copy of a randomly chosen firm that was profitable in the previous time step.

2.2. The Model

The model is composed of two vectors of agents: households,

$$H = [1, 2, \dots, n] \quad i \in H, \quad (1)$$

and firms,

$$F = [1, 2, \dots, m] \quad j \in F, \quad (2)$$

where each household (i) purchases q_i^j units from the firm (j_i^*), offering the lowest price known to it during time step t . All variables that are not exogenously set vary across time steps. For ease of explication, we will not include t as a subscript except when previous time steps ($t - 1$) are relevant.

Firms produce the primary good, Q_j , using inputs of labor, L_j , capital, K_j , and knowledge, A_j , where knowledge is composed of public, G , and private, R_j , knowledge:

$$\begin{aligned} Q_j &= A_j^\gamma K_j^\alpha L_j^\beta \quad \forall j, \\ A_j &= G + R_j, \end{aligned} \quad (3)$$

subject to the costs of production, C_j , including the wages, w_j , paid to employees; rent paid to capital, r ; and the investment in research and development, S_j :

$$\begin{aligned} C_j &= w_j L_j + r K_j + S_j \quad \forall j, \\ \pi_j &= p_j Q_j - C_j. \end{aligned} \quad (4)$$

Profits, π , are a function of Q_j sold at price p_j and C_j . Firms post unique prices in the market equal to lagged average cost ($AC_j = C_j/Q_j$) with a 50% markup,¹⁰

such that $p_{j,t} = 1.5 \cdot (AC_{j,t-1})$. Firms pay a wage equal to the lagged average marginal product of labor ($1/m \sum_j \text{MPL} \cdot p_{j,t-1}$).¹¹

During each step firms engage in research from which knowledge returns are uncertain, generating a quantity of private knowledge, or patent, $y_{j,t}$ that is temporarily excludable for Φ time steps, and contributes to a summed portfolio of private knowledge stocks, $R_{j,t} = \sum_{\phi=0}^{\Phi-1} y_{j,t-\phi}$. The process of research and development is modeled as an exponential probability function dependent on the firm's investment, S_j , its current portfolio of private knowledge, $R_{j,t}$, and the existing stock of public knowledge, G_t :

$$y_{j,t} = \left(\frac{-S_j}{G_t + R_{j,t-1}} \right) \log(Z) + R_{j,t-1} \quad \forall j, \quad (5)$$

where Z is a unit rectangular variate. The ratio of investment S_j to private knowledge created with each patent, $y_{j,t}$, is declining as the existing stock of knowledge, $G_t + R_{j,t-1}$, grows. This choice to model the costs of innovation as increasing with the existing stock of knowledge is based on the empirical observation that the costs of patents have been increasing over time [Kortum (1993)]. Firms choose unique research investments S_j equal to fraction χ of their total revenue from the previous turn, $\text{TR}_{j,t-1}$, adjusted by the factor $v_{j,t}$, where

$$\begin{aligned} S_{j,t} &= T R_{j,t-1} \cdot \chi_{j,t} \quad \forall j, \\ \chi_{j,t} &= \chi_{j,t-1} + v_{j,t}, \\ \text{where } \begin{cases} v_{j,t} = v_{j,t-1} & \text{if } \pi_{t-1} \geq \pi_{t-2} \\ v_{j,t} = -1 \cdot v_{j,t-1} & \text{if } \pi_{t-1} < \pi_{t-2} \end{cases} \end{aligned} \quad (6)$$

This research investment adjustment rule entails a simple profit-seeking heuristic on behalf of the firm, with which each individual firm gropes toward an investment procedure that increases profits. The increment of change, $v_{t=0}$, is an exogenously set parameter uniform across firms. Firms myopically grope toward greater profits, switching directions whenever their previous turn resulted in reduced profits.

Each firm's stock of private knowledge, $R_{j,t}$, is a rolling portfolio of patented knowledge. Each step, the oldest patent, $y_{j,t-\Phi}$, expires. The expired patent of greatest magnitude is added to the public knowledge stock, $G_t = G_{t-1} + \max\{y_{1,t-\Phi}, \dots, y_{m,t-\Phi}\}$. Research results in more efficient production that is rewarded by greater profits and greater prospects for long-run survival in the marketplace. This, in turn, incentivizes the long-run contribution to the public stock of knowledge and ideas in the form of expired patents, which lead to long-run growth. At the same time, the rolling expiration of patents allows turnover in private knowledge leadership at any given time step.

Household search occurs within each time step t in substeps $\tau = 1, \dots, m$, where each increment of τ represents an act of search by the agent.¹² Households search over the set of prices posted for q . Their search activities are governed by simple income-maximizing and cost-minimizing search functions based on

a desire to continue searching so long as the expected decreases in the lowest known price, p_i^* , will result in a net increase in purchasing capacity given the cost of an additional substep of search, $w_i \zeta$, where ζ is the amount of an agent's time endowment expended by an act of search. The decision variable is the number of search actions, τ , that constitute the fixed sample size that households decide on prior to the first discovered price.

Households assume a simple uniform distribution of prices $F(p)$ on $[\underline{p}, \bar{p}]$ and minimize the expected total cost (cost of purchasing q_{t-1} plus cost of search):

$$E(C_{i,\tau+1}) = q_{i,t-1} \left[\underline{p} + \int_{\underline{p}}^{\bar{p}} (1 - F(p))^\tau dp \right] + c\tau \quad \forall i. \quad (7)$$

For the sake of simplicity, and reasonableness as a heuristic accessible to agents with bounded rationality [Conlisk (1996)], households assume that the uniform distribution is bounded by prices equal to $2/3$ and $4/3$ of the price paid in the previous time step.¹³ The simple mathematical properties of a uniform distribution allow households to choose a fixed sample size equal to

$$\left(\frac{q_{i,t-1} \cdot \underline{p}}{c \cdot w_i} \right)^{\frac{1}{2}} - 1.$$

Each household i searches over the price set Ω , where $\Omega_{i,\tau}$ is the subset of prices known to household i after τ search efforts:

$$\begin{aligned} \Omega &\equiv \{p^1 \dots p^m\} \\ \Omega_{i,\tau} &\subset \Omega, \\ p_i^{j*} &= \min \left\{ p_i^j \mid p_i^j \in \Omega_{i,\tau} \right\} \quad \forall i. \end{aligned} \quad (8)$$

Once households have executed their searches and found a lowest known, they maximize a constant-elasticity-of-substitution (CES) utility function,

$$U_i = (q_i^\lambda + x_i^\lambda)^{1/\lambda}. \quad (9)$$

For a given wage rate and price, the optimal quantities of q and x are

$$\begin{aligned} q_i^* &= (1/p)^{1/1-\lambda} \cdot M / (p^{-\lambda/1-\lambda} + \eta^{-\lambda/1-\lambda}), \\ x_i^* &= (1/\eta)^{1/1-\lambda} \cdot M / (p^{-\lambda/1-\lambda} + \eta^{-\lambda/1-\lambda}), \end{aligned} \quad (10)$$

where the total income of the household, M_i , is a function of the household's wage, the number of substeps spent searching, the costs of search, ζ , and its dividend from capital rents and firm profits, d , such that $M_i = w_i(1 - \tau_i \zeta) + d$. The household wage, w , is a random draw from a lognormal distribution whose median is equal to the lagged average marginal product of labor.

The NTS acts as a single agent. It sets the price for x , η , based on the average cost of production from the previous time step, $\eta = C_{t-1}^{\text{NTS}} / (\sum_{h=1}^n x_h + \sum_{b=1}^z x_z)$. The NTS pays a wage to its employees equal to the average marginal product of labor within the set of firms from the previous step.

At the end of each step, all firms are evaluated for potential bankruptcy. All firms for which costs exceed revenues ($\pi_j < 0$) must borrow funds to remain solvent. This debt accumulates across steps. Bankruptcy occurs when accumulated debt exceeds the limit of B ,

$$B_j = \Gamma \cdot \max(\pi_{j,t=0} \dots \pi_{j,t}). \quad (11)$$

B is a function of the greatest profits previously realized by firm j in a single step, adjusted by an exogenous multiplier, Γ . A firm can borrow funds so long as its profit history indicates a reasonable expectation that they will be able to pay it off.

Entry in the model is handled with a relatively simple mechanism. A new firm enters the market whenever average profits are greater than zero. The new firm is a copy of a randomly chosen profitable firm, with the exception that it start without an existing stock of private knowledge.¹⁴

3. SIMULATION RESULTS

We ran the model under a variety of patent length and search cost parameterizations, with 200 time steps constituting a run. All simulations included 30,000 households and were initiated with 300 firms. Each combination of parameters was simulated 50 times, resulting in 8,250 total runs. The key exogenously set parameters are summarized in Table 1.

Our emphasis, in this paper, is on the distributional properties that are observable across firms. These properties, however, are of limited interest if they do not occur in a model of endogenous, exponential growth. Figure 1 plots $\log Q$, where $Q = \sum q_j$, over time in a single run of the model.

Tracking the growth of $\log Q$ over time, we observe two distinct periods. In the early time steps of the run, we see an “organizational” period in the model, which typically (but not always) concludes within the first 10 steps, within which growth is erratic, often characterized by large swings up and down, as firms grope toward profitable strategies. Eventually the model settles into a steady growth trend, while exhibiting small, irregular cycles that are indistinguishable from a random walk. Growth observed in the model is exponential and consistent.

Outside of our primary experimental parameters, patent length and search cost, there are two notable sensitivities in the model. The first sensitivity is to our bankruptcy parameter—specifically, the limit to the net losses that a firm can bear before it goes bankrupt. Given the competitive nature of the market and the ability of firms with superior technology to dominate, if Γ is sufficiently low, the market will always collapse to monopoly.¹⁵ This tendency toward monopoly when lagging firms have weaker access to loanable funds is an expected outcome and,

TABLE 1. Model parameters

Parameter	Context/related function	Value
M	Starting number of firms	300
N	Number of households	30,000
α, β	$Q_j = (G + R_j)^\gamma K_j^\alpha L_j^\beta$	0.5
γ	$Q_j = (G + R_j)^\gamma K_j^\alpha L_j^\beta$	0.1
λ	$U_i = (q_i^\lambda + x_i^\lambda)^{1/\lambda}$	-0.1
$G_{t=0}$	Initial public stock	10
Γ	Loanable funds multiplier	3
Ψ	$r(Q_{t-1}, \Psi)$	0.01
$v_{j,t=0}$	$\chi_{j,t} = \chi_{j,t-1} + v_{j,t}$	0.001
ζ^\dagger	Search cost	[0.005, 0.5]
	Patent length	[2, 50]

[†]The total substep time endowment for an agent is 1. Thus, when search costs, ζ , equal 0.05, that is equivalent of 5% of their time endowment, meaning it takes 5% of an agent’s substep time endowment to engage in another act of search. The exponents on labor (L) and capital (K) in the NTS are 0.4, leading to diminishing returns to scale.

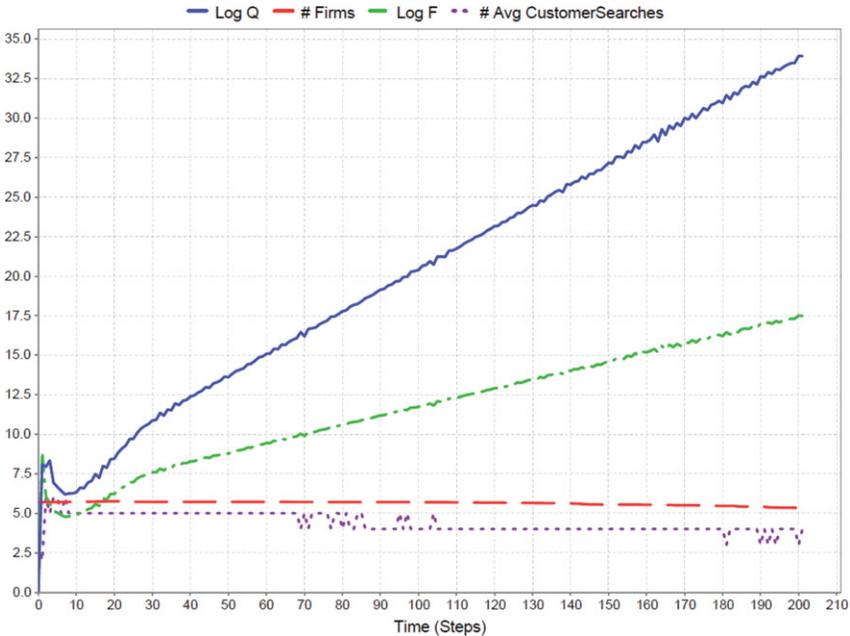


FIGURE 1. Key metrics over the life of a single run, including $\text{Log } Q$, $\text{Log } F$, the number of surviving firms, and the average number of customer searches conducted by households. Patent length = 20, search costs = 1.5%. The model was initiated with 30,000 households and 300 firms.

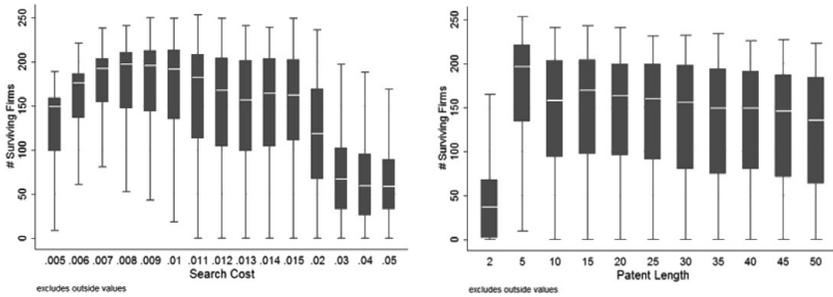


FIGURE 2. Surviving firms over (left) search cost and (right) patent length.

in turn, a limitation on the range of testable parameters. We parameterized our experiments with a value of $\Gamma = 3$ both because it is reasonable to imagine a firm will be able to secure funding that its personal history suggests it can pay back in fewer than three time steps and because it leads to a rate of market collapse that is sufficiently low across our experiments. The second sensitivity in the model lies in the output elasticities assigned to labor, capital, and the accessible stock of technology. Patterns of growth over time become increasingly unstable as output elasticities are reduced. We choose relatively straightforward output elasticities, creating diminishing returns to scale in the NTS and increasing returns to scale in technology-enabled production.

3.1. Market Outcomes

Firm survival in our model is predicated on the notion that although the firm with the leading technology will always have a considerable advantage in terms of profit margins and attracting customers with low prices, technologically inferior firms can survive because costly search limits customer information. Thus, we expect that the number of surviving firms after 200 time steps will increase with search costs. In Figure 2, we construct box and leaf plots of the distribution of surviving firms across each search cost tested. Starting from our lowest search cost, the number of surviving firms quickly rises.¹⁶ However, it proceeds to drop off after search cost exceeds 1%. It would appear that as search costs become too high, shrinking customer search activity leads to an increased firm failure rate as firms run the risk of not being found by enough customers, regardless of how low their prices may be.

The number of surviving firms steadily decreases with the length of patents (Figure 3, right panel). This follows basic intuition—the longer patents last, the less frequent the turnover in relative technological ordering, and the bigger the advantage that leading firms can generate and, in turn, use to drive a larger number of competitors into bankruptcy. That said, patent length of 2 steps produces fewer surviving firms than lengths of 5 through 30 steps. One possible explanation is

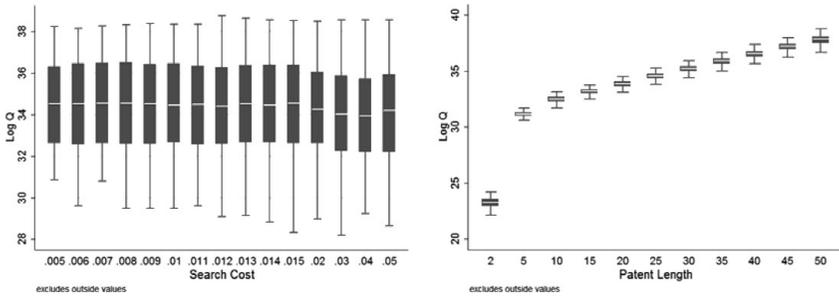


FIGURE 3. Log Q over (left) search cost and (right) patent length.

that such short patent lengths lead to so much noise in the model that there exists no long-term strategy that can sustain a large number of firms.

In terms of total market productivity, we do not observe the expected frictional loss from search costs. In the left panel of Figure 3 we can see that $\log Q$ is essentially indistinguishable across search costs, with a small decline in the mean with the largest values. In contrast, we can see in the right panel of Figure 3 that $\log Q$ is steadily rising with increased patent length. The gains to increased patent length are sharpest early on, but they remain steadily positive across our entire range of tested values.

In a simple OLS regression of $\log Q$ on patent length and search cost, search cost has a negative (albeit relatively small) coefficient and patent length has a positive coefficient.¹⁷ Given the randomness of research outcomes and firm survival rates, we can include the number of surviving firms as a control variable. The coefficient signs remain the same, but it worth noting that when patent length and search cost parameterizations are controlled for, $\log Q$ is positively correlated with the number of surviving firms.

Despite the existence of increasing returns to scale, and the possibility of a Pareto-optimal natural monopoly, markets characterized by patent protection that remain competitive lead to superior outcomes through their motivation of greater research investments across firms and the larger number of random draws from the distribution of research outcomes. This result is similar to that in Judd (1985). Models exploring optimal patent length come to a variety of conclusions, some emphasizing the tendency toward growth increasing with patent protection [Kwan and Lai (2003)], whereas others emphasize the presence of an internal (finite) peak [Horowitz and Lai (1996); Iwaisako and Futagami (2007)]. There is also a growing empirical literature that argues that patent protection may be too strong (or at least too blunt an instrument) in the United States [Gallini (2002)] and that growth correlates negatively with patent length [Dosi et al. (2006); Boldrin and Levine (2008, 2013)]. Given that there are (a) always increasing returns to scale in our model, (b) zero frictional loss to patent protection, and (c) perfect sharing of knowledge upon patent expiration, and that (d) patents never act as a barrier to

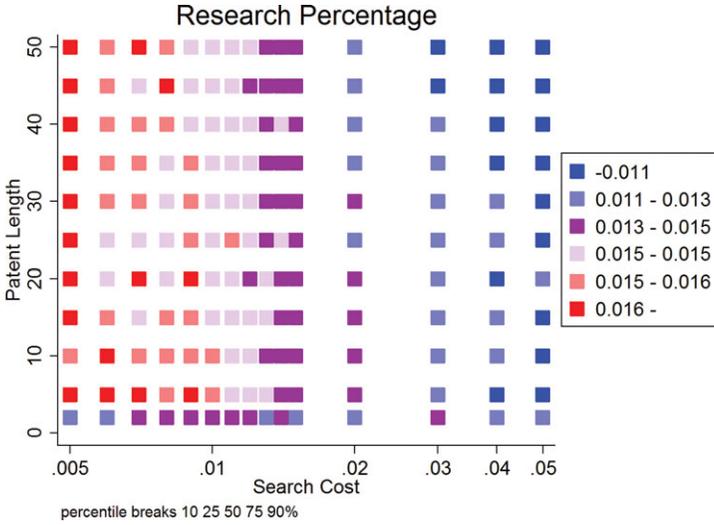


FIGURE 4. Heat map of research investment percentage over search cost and patent length.

innovation (e.g., there are no obstructive patents or “patent thickets”), the positive relationship with growth is not surprising. Future work looking to model the role of patents in overall growth would require extending the model to incorporate a richer construction of intellectual property legal institutions and concomitant firm strategies.

3.2. Outcomes across Firms

Firms in our model are homogeneous *ex ante* and heterogeneous *ex post*. Given the randomness of research outcomes and cost constraints faced by searching households, each firm experiences its own unique history of research outcomes, sales, and profitability. Firms are confronted with two forms of uncertainty: research uncertainty and commercial uncertainty.¹⁸ They do not know the stock of excludable knowledge that their research investment will bear, nor do they know whether customers will successfully find them *even if* they are able offer a relatively low price. These uncertainties result in differing research investments, private knowledge stocks, posted prices, q_j sold, and revenues generated.

Research investment is the manner in which the individual firms most directly respond to their own unique history. In Figure 4 we can see a heatmap of the mean research investment percentages across different combinations of patent length and search cost. The average research percentage is clearly decreasing with search costs. It is perhaps surprising that there is no immediately observable relationship between patent length and research investment percentage.

Beyond just the mean, changes in the shape of the research investment percentage distributions are notable. In Figure 5 we present the histograms of firm

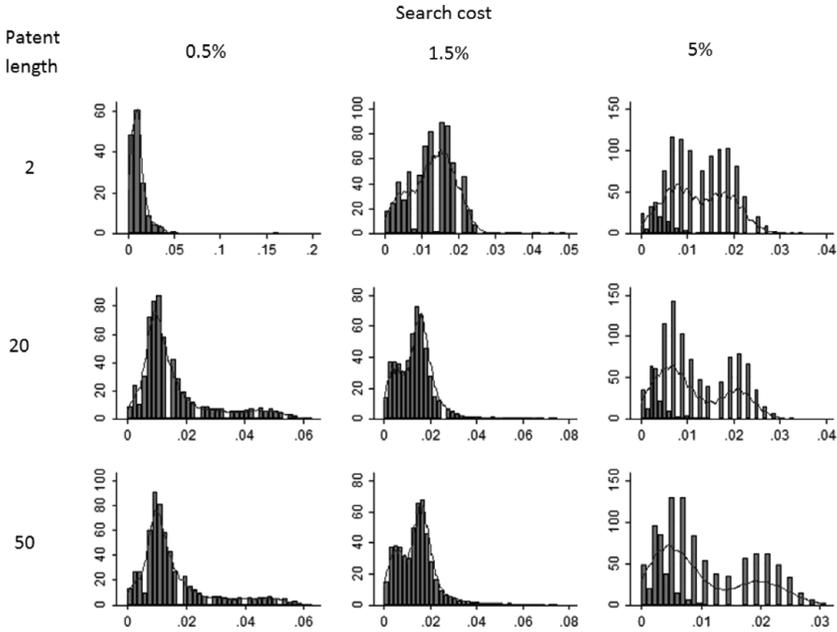


FIGURE 5. Distribution of research investment percentage, organized by patent length (2 to 50, vertically) and search cost (1%, 1.5%, and 5%, horizontally), at $t = 200$. Each subfigure is a histogram of research investment percentage from surviving firms across 50 runs of the model. Note that the subfigures do not have identical ranges on their x axes.

research investment percentages for a representative sample of combinations of search cost and patent length parameterization. We observe steadily bifurcating distributions as we increase search costs, with distinct “high research” and “low research” modes emerging. A similar bifurcation of research strategies has been identified empirically [Hoskisson and Johnson (1992); Lee and Harrison (2011)]. At the same time, we see the upper tail of the distribution of research percentages stretching out with greater patent lengths.

In much of the analysis to follow, we are mindful of scale effects, given that some runs of the model will produce higher growth rates that compound over the 200 steps, resulting in considerably larger total revenues across firms. When we are concerned with the shape of the revenue distribution, we scale total firm revenue, TR_j , to the individual run of the model by dividing TR_j by the mean total revenue within each run of the model.¹⁹

Empirical research has found a pattern of fat tails, including Pareto–Levy distributed upper tails, in the distributions of firm size and the returns to patents. Some industry distributions, specifically pharmaceuticals, have also been identified as distinctly bimodal. Figure 6 presents histograms of logged total revenue (scaled), across all firms at time step 200, parameterized with different patent

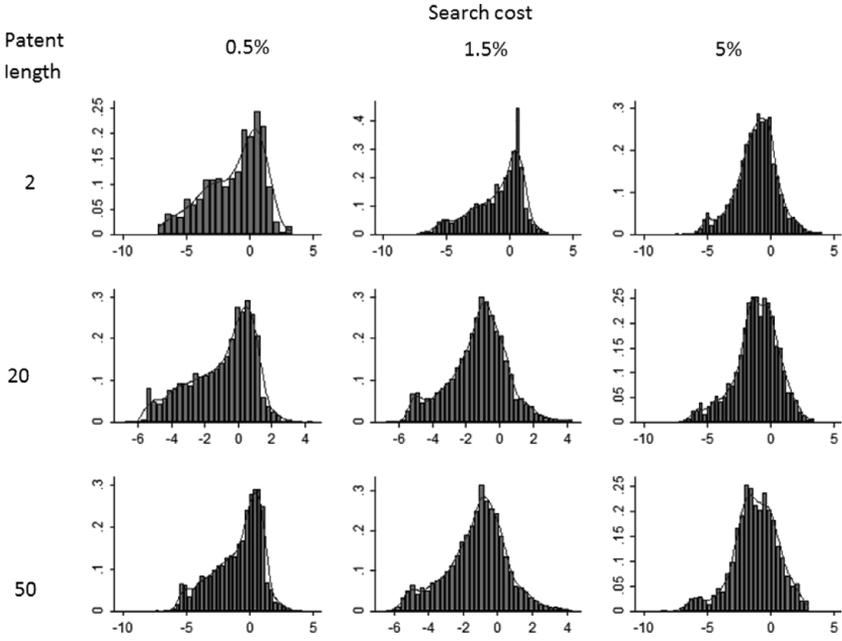


FIGURE 6. Distribution of log total revenue (scaled to the mean) organized by patent length (2, 20, and 50, vertically) and search cost (1%, 1.5%, and 5%, horizontally) at $t = 200$. Each subfigure is the distribution of log total revenue across firms from 50 runs of the model. Note that the subfigures do not have identical ranges on their x axes.

lengths and search costs (each panel represents data from 50 runs of the model). Model outcome distributions have consistently fat tails (see Table 2). Although a traditional bifurcation can be observed visually in the research percentage as search costs increase, bimodality is immediately visually apparent, as in the total revenue results. However, there is discussion in the literature of a subtler “core vs. fringe firms” bimodality of the firm size distribution [Bottazzi et al. (2007); Mazzucato and Demirel (2007); Dosi and Nelson (2010)]. To test for the presence of two underlying distributions, we employ a simple test of statistical bimodality. The SAS User’s Guide [SAS Institute (1989)] includes a coefficient of bimodality, essentially kurtosis compensated for skew, that has been used in studies of dimorphism in the biological sciences [Ellison (1987); Imbert et al. (1996)]:

$$\text{Bimodality} = \frac{\text{skew}^2 + 1}{\text{raw kurtosis}}. \tag{12}$$

If $\text{Bimodality} > 0.55$, a distribution’s bimodality is deemed significant. Across the entire body of simulation runs, 83% of distributions of TR_j and 57% of distributions of research investment percentages exceeded the bimodality

TABLE 2. Log TR: Kurtosis and Pareto–Levy OLS regression coefficients at $t = 200$

Patent length	Search cost	(1)	(2)	(3)	(4)	(5)
		Mean kurtosis ^a	OLS $ \varpi $	OLS $ \varpi $ 4th quartile	OLS $ \varpi $ 4th quartile†	Hill $ \varpi $ 4th quartile ^b
2	0.5%	10.90	0.308	0.852	1.688	1.76
2	1.5%	31.77	0.360	1.299	1.945	1.95
2	5%	37.48	0.513	0.971	1.343	1.17
20	0.5%	39.31	0.393	1.331	2.123	1.8
20	1.5%	77.93	0.476	0.851	0.961	0.93
20	5%	16.78	0.441	0.969	1.360	1.20
50	0.5%	52.91	0.394	1.397	1.472	1.43
50	1.5%	58.31	0.473	0.867	0.826	0.76
50	5%	13.37	0.426	0.862	1.472	1.35

^aColumn 1 is the raw kurtosis (kurtosis = 3.0 in a normal distribution) of distribution TR_j (scaled) in each run of the model (i.e., not the logged values).

^bColumns (4) and (5) use data only from the run of the model with the largest number of surviving firms for that combination of parameter values.

threshold. The results for runs with more than 20 surviving firms are summarized in Table 3. Although the distribution of research investments was obviously visually bifurcated with high search costs, we see that the distributions with an extended right tail associated with lower search costs is being identified as a second mode as well. Nearly every run of model that remains decentralized (e.g., with more than 20 surviving firms) has a bimodality coefficient of TR_j that exceeds the test threshold.

In Figure 7 we chart the rank, N , of each observation, where the rank can be interpreted as the number of other observations within the same model simulation run that are of equal or greater value to $\log TR_j$ at step $t = 200$.²⁰ The shape of the results in Figure 7 resembles what was found by Scherer et al. (2000) in their study of the value (in Deutschmarks) of German patents from 1977 to 1995, duplicated here in Figure 8. In both the simulation and patent data, the lower-value observations within the distribution are concave to the origin, but the higher-value observations have a more linear relationship between log value and log rank. A linear relationship between the log of the rank and the log of the value is indicative of a power law distribution. Both the simulation results and German patent return observations also include outliers significantly beyond the rest of the distribution.

Scherer et al. (2000) analyze the Pareto–Levy, or power law, distribution parameters of the patent data with simple ordinary least squares regression analysis using the log-linear modeling function

$$N = k(TR_j)^{-\varpi},$$

$$\log N = \log k - \varpi \log TR_j, \quad (13)$$

TABLE 3. Bimodality of research percentage and total revenue (scaled)

Patent length	Search cost	Research percentage		Scaled total revenue	
		(1) Mean Bimodality	(2) % > 0.55	(3) Mean Bimodality	(4) % > 0.55
2	0.5%	0.59	51%	0.82	100%
2	1.5%	0.47	26%	0.74	93%
2	5.0%	0.52	31%	0.85	100%
20	0.5%	0.72	98%	0.73	92%
20	1.5%	0.43	8%	0.87	100%
20	5.0%	0.68	91%	0.83	100%
50	0.5%	0.71	98%	0.72	95%
50	1.5%	0.45	10%	0.87	100%
50	5.0%	0.69	95%	0.84	100%

Notes: Total firm revenue is scaled by the mean within each run. Each value is calculated across 50 runs of the model. Table includes only runs with more than 20 surviving firms. The results are largely the same when all runs are included, but the bimodality index is less informative with sparsely populated distributions.

where N is the rank of the return on investment observation, TR_j is total revenue, and k and ω are parameters. Absolute values of ω greater than or equal to one are indicative of a Pareto–Levy distribution.

In most of their patent data, Scherer et al. do not find a log linear fit between TR_j and rank across the full body of observations, instead finding that the bulk of the distribution is better characterized as lognormal. However, they do observe a much closer to linear fit in the upper tail of the data. We find similar results in our simulation data across a variety of parameterizations. For each run of the model we rank the firms according to revenue size. Simple ordinary least squares analysis of Pareto–Levy model parameters, when regressed over the full distribution of $\log TR_j$, results in ω slope coefficients less than one. However, if we isolate the third and fourth quartiles of the distributions, the resulting coefficients correspond to a Pareto–Levy distribution. The results of the ω slope parameter are included in Table 2 for the overall distribution [column (2)], and in the fourth quartile [columns (3) and (4)] for each combination of patent length and search cost parameter combination tested. In all of the model parameter combinations, the regression of Pareto–Levy parameters on the full distribution resulted in slope coefficients less than one. Regression on the fourth quartile observations resulted in much larger slope coefficients, with $|\omega|$ near 1. Regression on the fourth quartile in only the runs of the model with the highest survival rates results in $|\omega|$ near or exceeding 1 in all but one of the parameter combinations. In column (5) of Table 2 we report the results of using the Hill estimator [Hill (1975); Scotto (2001)] to

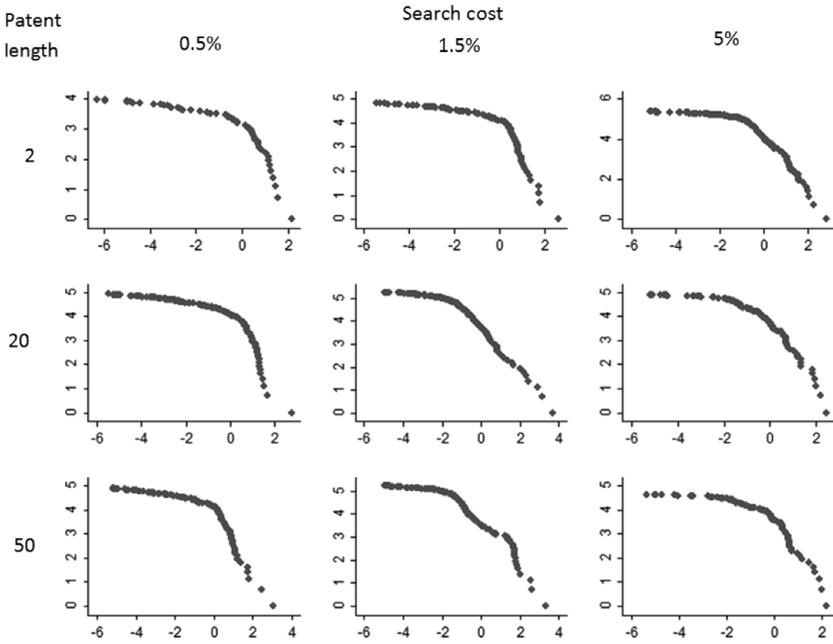


FIGURE 7. Log rank_j over log TR_j (scaled to the mean) organized by patent length (2, 20, and 50, vertically) and search cost (0.5%, 1.5%, and 5%, horizontally) at $t = 200$. The run from each set of 50 simulations with the largest number of remaining firms is portrayed. Values averaged across all 50 runs are plotted in the Appendix.

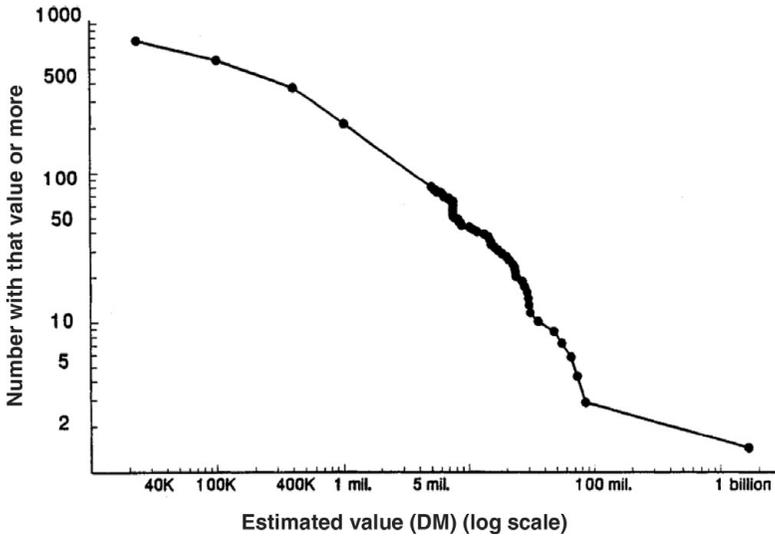


FIGURE 8. Rank over estimated value (log–log scale), from Scherer et al. (2000).

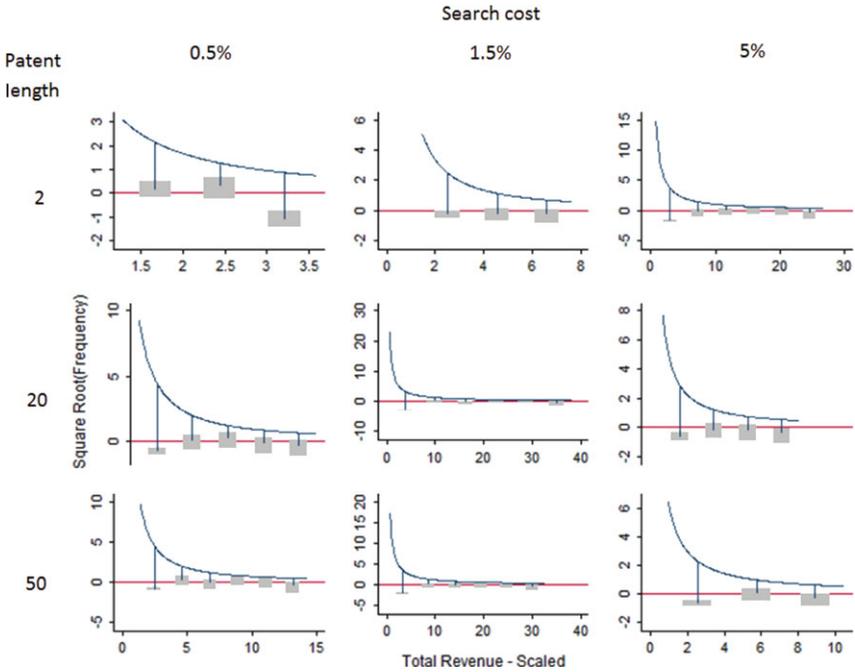


FIGURE 9. Hanging rootograms of the distributions of total revenue (scaled) organized by patent length (2, 20, and 50, vertically) and search cost (0.5%, 1.5%, and 5%, horizontally) at $t = 200$. Each subfigure uses a distribution of model-generated firm growth rates from surviving firms across 50 runs of the model.

estimate the value of ω . We run the Hill estimator only on the run from each parameter combination that has the highest firm survival rate. The results further support the simpler OLS analysis.

Additionally, in Figure 9 we present *hanging rootograms* [Tukey (1972, 1977)] comparing the model upper quartile of the distribution of firm revenues (scaled by the mean firm) to the hypothesized Pareto distribution growth rate distributions with the hypothesized distribution. The predicted values of the hypothesized distribution are drawn, with model generated histogram bins “hanging” from it. The y axis is scaled to the square root of the frequencies to emphasize discrepancies in the tails of the distribution. A perfect fit of a section of the model distribution to the fitted Pareto distribution would result in the hanging bar perfectly touching the horizontal bar at zero. In each subfigure, 95% confidence intervals are represented by gray rectangles in each. As befits the results in Table 2, the model consistently produces upper tail values that fit the Pareto distribution at the 95% confidence level.

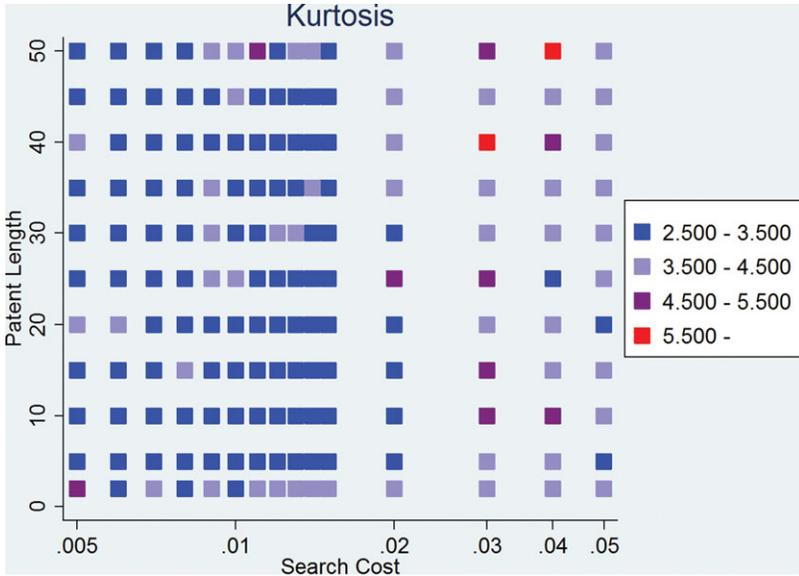


FIGURE 10. Heat map of growth rate kurtosis over search cost and patent length.

3.3. Firm and Market Growth

There exists a literature identifying the growth rate of firms as Laplace (double exponential) distributed [Bottazzi and Secchi (2003, 2006)]. These distributions are characterized as “tent-shaped,” with fatter than Gaussian tails. We calculated the growth rate of a firm j within our model as

$$\text{growth}_j = \log(\text{TR}_{j,t=200}) - \log(\text{TR}_{j,t=150}). \tag{14}$$

We can see in Table 4 that, using the Doornik–Hansen (2008) test, we can reject normality in the growth rate distributions²¹ and that distribution tails are fatter than Gaussian with the highest search costs. In Figure 10 we summarize the kurtosis of the distribution of growth using a heat map of outcomes across patent length and search cost. Tail mass is increasing with search cost across all patent lengths, though it rarely exceeds 6 (the kurtosis of a numerical Laplace distribution). To test whether they are properly characterized as Laplace distributed, we again constructed hanging rootograms comparing the model-generated growth rate distributions with the hypothesized distribution (Figure 11). Deviation from the Laplace distribution is represented by the deviation from the hanging roots from the red baseline, with 95% confidence intervals again represented with gray rectangles in each subfigure. As expected given the kurtosis results in Figure 10,

TABLE 4. Market and firm growth distributions: Kurtosis

Patent length	Search cost	Firm growth kurtosis	Doornik–Hansen test ^a
2	0.5%	5.14	$p < 0.001$
2	1.5%	3.54	$p < 0.001$
2	4%	3.84	$p < 0.001$
2	5%	4.13	$P < 0.001$
20	0.5%	3.54	$p < 0.001$
20	1.5%	3.13	$p = 0.130$
20	4%	4.33	$p < 0.001$
20	5%	3.49	$p < 0.001$
50	0.5%	2.97	$p = 0.068$
50	1.5%	3.31	$p < 0.001$
50	4%	6.78	$p < 0.001$
50	5%	4.19	$p < 0.001$

^aThe Doornik–Hansen (2008) test has a null hypothesis of normality. We can reject normality at the 0.1% level in all of our log growth distributions.

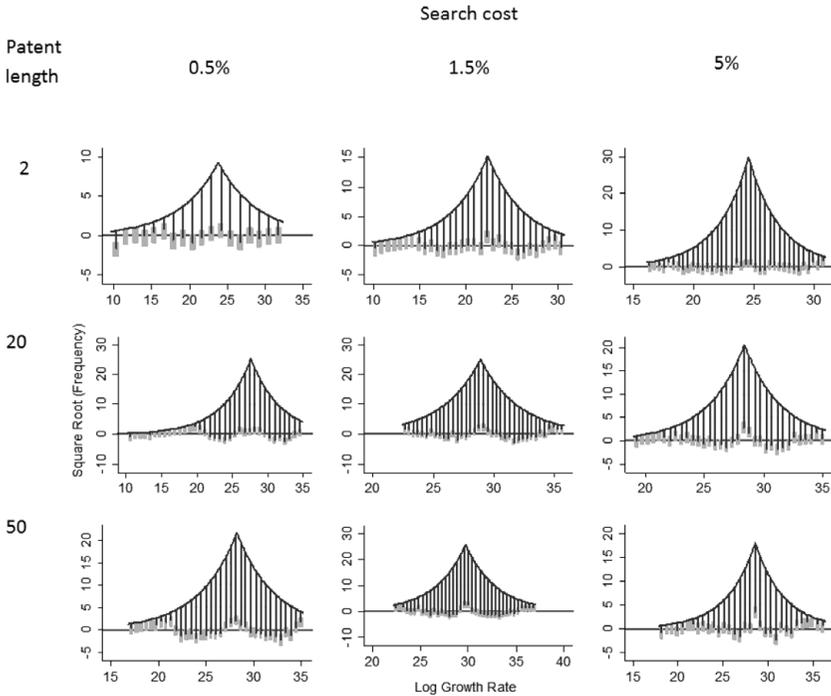


FIGURE 11. Hanging rootograms, firm growth distributions, organized by patent length (2, 20, and 50, vertically) and search cost (0.5%, 1.5%, and 5%, horizontally) at $t = 200$. Each subfigure fits the Laplace distribution to the distribution of model-generated firm growth rates from surviving firms across 50 runs of the model.

the Laplace distribution is a better fit with higher search costs, though it arguably remains a passable fit in all of the parameter combinations.

It is our view that, considered together, the Pareto–Levy coefficients and general visual shape of the log rank–log TR_j data plots, the bimodality of TR_j and research percentage, the high kurtosis values of TR_j (Table 2), and the Laplace distribution of firm growth rates are evidence that the model has emerging distributional patterns that match the characteristics that are associated with patents and research-dependent industries.

4. DISCUSSION

There are a variety of stylized facts associated with the distributional characteristics of firms and the returns to innovation. Some, such as power-law distributed upper tails, have been associated with the presence of increasing returns to scale [Axtell (1999)]. Others, such as bimodal research investment strategies, are associated with leader and follower dynamics in competitive markets [Hoskisson and Johnson (1992); Lee and Harrison (2011)]. We believe that the reason that simultaneous replication of these stylized facts within a model has proven elusive is the difficulty in maintaining a competitive market in a model with increasing returns to scale. It is our view that the model is producing results characterized by a disparate set of distributional patterns that match the peculiar characteristics associated with patents and research-dependent industries because it manages to maintain a competitive market while all firms benefit from increasing returns to scale. Thus, our model represents a strong complement to recent work characterized by either imperfect competition or constant returns to scale [Dosi et al. (2010); Dawid et al. (2011); Dosi et al. (2012)]. For comparison, Dosi et al. (2010) build an agent-based model characterized by both constant returns to scale and imperfect competition that generates fat-tailed growth rates and skewed firm size distributions, but does not, however, generate bimodality or Pareto distributed upper tails.

The distribution of returns to research investment, in our model, appears to be bimodal, fat-tailed, and dominated by Pareto–Levy power-law distributed upper quartiles. None of the component distributions of this curious mixture are assumed in the model. The economic consequences of perfect substitution combined with positive search costs allow an occasional innovation to revolutionize the industry without collapsing the model to a single firm. Such revolutions happen repeatedly, over time. The more a firm grows, the more it can invest in research. Although any firm can, in theory, be the technological leader of tomorrow, the current leader always has a built-in advantage. This advantage can compound over time, dependent on the patent length, leading to the observation of outliers. If a firm gets lucky enough times in a row, it can put a significant gap between itself and the rest of the marketplace.

If at the same time such a fortunate firm chooses to make a significant investment in R&D, this coincidence of fortune and investment creates the potential for both enormous firm gains in the short run and enormous societal gains in the long run. If a firm does not find itself benefiting from past wagers on research outcomes, however, it may drift toward lower levels of research investment (as a percentage of revenue). It is in this manner that we observe the distribution of research strategies bifurcating toward low and high percentages, generating something akin to the R&D leaders and followers model. The distributions observed in Figures 6 and 7 are a competitive outcome of the interaction of a probabilistic dynamic (how many times in a row can a firm get a fortunate return on investment in innovation relative to other firms?) and a decision made by firms with limited information that slowly bifurcates the landscape (will the firm invest more or invest less in R&D?).

How might one describe a minimal axiomatic account of an evolutionary process? The first step is well known. Rules of behavior are laid down that can then be used to understand the probability of some characteristic's survival. This is a procedure that is invariably followed in evolutionary economic modeling [Nelson and Winter (2002)], ours included. But is that all there is? Can we observe the "origin of species"? The 19th-century statistical biologists, some of whom would become founding editors of *Biometrika*, took a neglected second step. We quote from a wonderful recent history of the episode:

Weldon's observation of a double-humped distribution in the breadth of the foreheads of Naples crabs . . . gave him and Pearson hope that they were gaining a glimpse of the bifurcation of a species by natural selection. If this pattern became more exaggerated over many generations, eventually there would be two separate species. Alfred Russel Wallace had predicted that evidence of natural selection would be found in empirical plots that deviated from a bell curve. [Klein (1997, p. 178)]

The technical problem of describing a plausible distribution for Pearson's other species, prawns, was magnified because of the presence of "giants" which were "in some degree anomalous" [Pearson (1894, p. 101)]. For Pearson, Wallace, and Weldon, the statistical signature of the evolutionary process is the emergence of abnormality. We have followed a two-part procedure, first describing the rules that our agents follow, and then examining the distribution of characteristics that results from our simulation of an evolutionary process. Following Weldon and Pearson, our observation of bimodal distributions with extreme outliers can be thought of as evidence of "type-endogeneity" and a signature of evolutionary processes.²²

Although phylogenetic divergence is an intuitive evolutionary story, Pareto–Levy upper tails are similarly part of the evolutionary oeuvre of models, specifically with regard to technical change [Dosi and Nelson (2010)], a frequent example being the evolution of computing power over time [Nordhaus (2007)]. Although our model is not explicitly within the evolutionary modeling tradition, it produces a variety of stylized facts associated almost exclusively with evolutionary and Schumpeterian models.

5. CONCLUDING REMARKS

Grossly simplified, to generate exponential growth over time, a model must typically have a source of increasing returns to scale. To generate bimodality, in both research strategy and returns to innovation, a model must have heterogeneous agents and remain competitive. To generate power law upper tails and outliers, a model must be dynamic and allow innovation and success to compound over time. It would be incredibly difficult to generate all of these properties, as well as others such as Laplace distributed firm growth, within a general equilibrium framework. Our model is able to exhibit all three model characteristics (increasing returns, competition, innovation) by building costly search and excludable knowledge (patents) into the model, allowing lagging firms to cope with their inferior technology while hoping for better future outcomes. We believe that the model's disequilibrium nature proves quite useful in this context.

It would be useful to extend and test the model in a computing environment that allowed larger-scale simulations. Given the importance of highly skewed distributions, larger agent and firm pools could have important ramifications for growth rates and the incentives to participate in innovation races. Patents in our model are greatly simplified. Future work would benefit from introducing more sophisticated intellectual property rights, including both "breadth" and length, as well as a continuum of imitation and obsolescence. This paper is largely concerned with offering a process to generate the unusual distributional characteristics of the returns to research and the need for their realization in endogenous technical change models. More generally, the nature of the "optimal" patent length is given only cursory attention here, and would benefit from finer grain analysis in future work.

NOTES

1. To varying degrees, the models proposed in this literature are built using the foundations laid out by Judd (1985), Grossman and Helpman (1991), and Aghion and Howitt (1992).

2. Luttmer (2007) is noteworthy for building a model of endogenous growth that is consistent with the observed Zipf distribution of firm size.

3. In the interest of clarity, our model is not perfectly competitive. Given that information is costly and prices are above average costs, competition in our model can be qualified as "imperfect."

4. Mixed distributions with extreme values have also been offered as a tractable representation of Knightian uncertainty and a challenging environment for policy [Epstein and Wang (1994)].

5. Models are often built with a single final good, but a continuum of intermediate goods [Grossman and Helpman (1991)]. Models of Schumpeterian growth often introduce Bertrand competition into the model by assuming a duopolistic (leader and follower) structure where the two firms compete through R&D investments. For a summary of the Schumpeterian growth literature, see Aghion et al. (2013).

6. By "deep heterogeneity" we mean that each agent is in principle unique. Economic models will often have a handful of agent "types," but thousands of unique agents would not be mathematically tractable.

7. For a discussion of scale-dependent vs. scale-independent models of endogenous growth, see Eicher and Turnovsky (1999).

8. In contrast to the bulk of the existing literature, Hellwig and Irme (2001) build a general equilibrium model of endogenous technical change that includes competitive markets, though their unique equilibrium is characterized by a low steady-state growth rate.

9. The model is written in Java using the MASON agent modeling library [Luke et al. (2005)]. The step/substep construct is built into the MASON model scheduling system.

10. Allowing a markup of price above average cost is a hallmark of imperfect competition, though there is no guarantee for any firm in our model that they will be able to attract customers with their price.

11. The marginal product of labor is $A_j \beta K_{j,t-1}^{\alpha} L_{j,t-1}^{\beta-1} \cdot P_{j,t-1}$.

12. There are m firms, and thus m prices over which to potentially search. If the cost of a unit of search, Δh , equaled zero, all agents would continue to search until τ equaled m .

13. We tested the model with multiple distributional assumptions and search rules, including log-normal and exponential distributions of varying precision, and perfect knowledge of the distributional boundaries. All search rules produced qualitatively similar results.

14. We tested two alternative entry regimes as well, (1) creating a new firm when there existed *any* profitable firms and (2) no entry at all. Although there are differences in the number of surviving firms at the end of the model run, the distributional properties remain qualitatively the same.

15. The model can also collapse to zero firms—with all going bankrupt. This is a result of technological and competitive uncertainty, imperfect heuristics, and increasing returns to scale. Multiple firms sometimes find themselves splitting a market they previously dominated, only to all be producing at a level that does not allow them to cover costs.

16. It is not shown here, but the model almost universally collapses when search costs are set to 0.1%.

17. The coefficients are statistically significant at the $p < 0.001$ level, but statistical significance is, for the most part, a trivial component of our results. We have an extremely large sample size, and standard deviations are relatively small, because only two parameters are manipulated in our model. We are, for the most part, concerned with coefficient magnitudes, not standard errors.

18. This is not unlike the three types of uncertainty (technical, commercial, and financial) laid out in Scherer et al. (2000).

19. For example, a TR_j value of 3.8 would mean that the firm j had total revenue 3.8 times as large as the mean firm in the population of surviving firms in that run of the model.

20. For ease of viewing, we only include the plots for the run, within each combination of parameters, that resulted in the most surviving firms. In the Appendix we include a similar figure charting the mean log TR_j for each associated rank within 50 simulation runs with each combination of parameter values. The figures are qualitatively similar, though there is noise from the different growth patterns across runs.

21. One combination, patent length of 20 and search cost of 1.5%, did fail to reject normality ($p = 0.13$), but taken as a whole we are comfortable rejecting normality in the growth distributions.

22. Type-endogeneity has been put forward as a necessary and sufficient condition for race-blind models of types of the sort that descend from Adam Smith [Levy and Peart (2008)].

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APPENDIX

In Figure A.1 we chart the rank, N , of each observation, where the rank can be interpreted as the number of other observations within the same model simulation run that are of equal or greater value than $\log TR_j$ at step $t = 200$. The x axis is the mean $\log TR_j$ (scaled to the mean of each individual run) across the 50 runs with each parameter combination, for each rank. The shape of the results presented here is similar to that of those in Figure 7. The figure is flattened and skewed downward at the highest ranks by model runs that collapse to small numbers (the highest rank values are especially skewed by runs that converged to monopoly). Figure A.1, when compared with Fig. 8 in the body of the test, serves to demonstrate that the basic shape of plots is largely consistent across runs.

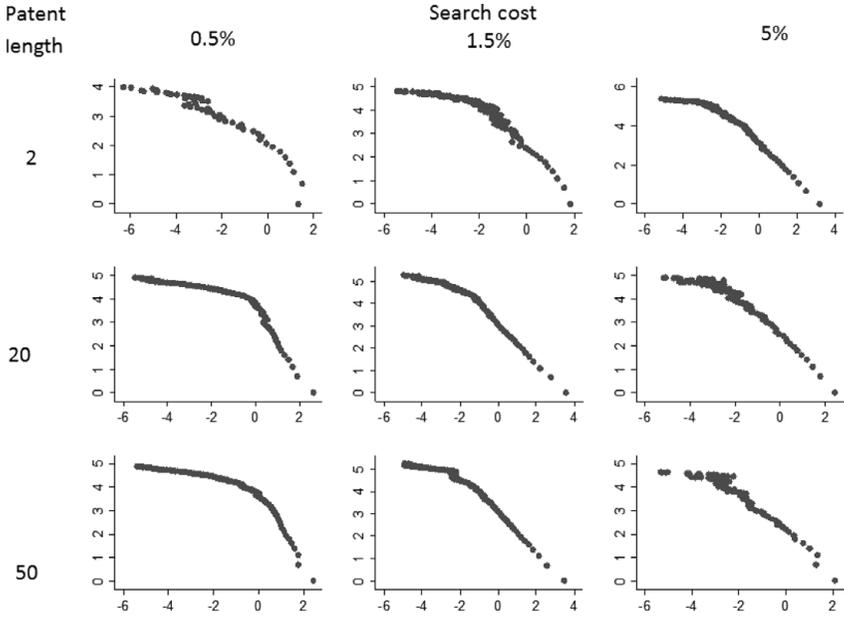


FIGURE A.1. Mean $\log \text{Rank}_j$ over mean scaled $\log \text{TR}_j$ organized by patent length (2, 20, and 50, vertically) and search cost (0.5%, 1.5%, and 5%, horizontally) at $t = 200$. The mean of the 50 runs from each combination of parameter values is portrayed.